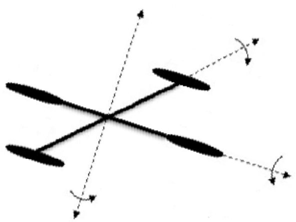
Laplace Transform -4



Additional Operational Properties of Laplace Transform

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}[e^{-st} f(t)]\,dt = -\int_0^\infty e^{-st}\,tf(t)\,dt = -\mathscr{L}\{tf(t)\};$$

that is, $$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\mathscr{L}\{f(t)\}.$$

Similarly, $$\mathscr{L}\{t^2 f(t)\} = \mathscr{L}\{t \cdot tf(t)\} = -\frac{d}{ds}\mathscr{L}\{tf(t)\}$$

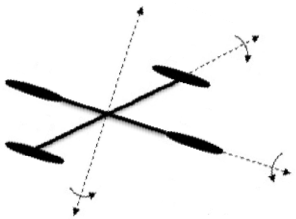
$$= -\frac{d}{ds}\left(-\frac{d}{ds}\mathscr{L}\{f(t)\}\right) = \frac{d^2}{ds^2}\mathscr{L}\{f(t)\}.$$

The preceding two cases suggest the general result for $\mathscr{L}\{t^n f(t)\}$.

THEOREM 4.8 Derivatives of Transforms

If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots,$ then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$



Example 1 **Using Theorem 4.8**

Evaluate $\mathscr{L}\{t \sin kt\}$.

SOLUTION With $f(t) = \sin kt$, $F(s) = k/(s^2 + k^2)$, and $n = 1$, Theorem 4.8 gives

$$\mathscr{L}\{t \sin kt\} = -\frac{d}{ds}\mathscr{L}\{\sin kt\} = -\frac{d}{ds}\left(\frac{k}{s^2 + k^2}\right) = \frac{2ks}{(s^2 + k^2)^2}.$$ ❑

Example 2 **An Initial-Value Problem**

Solve $x'' + 16x = \cos 4t, \quad x(0) = 0, \quad x'(0) = 1.$

of 1 foot per second in the downward direction from the equilibrium position.

Transforming the differential equation gives

$$(s^2 + 16)X(s) = 1 + \frac{s}{s^2 + 16} \quad \text{or} \quad X(s) = \frac{1}{s^2 + 16} + \frac{s}{(s^2 + 16)^2}.$$

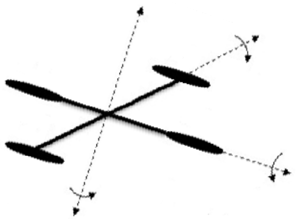
Now we have just learned in Example 1 that

$$\mathscr{L}^{-1}\left\{\frac{2ks}{(s^2 + k^2)^2}\right\} = t \sin kt, \tag{1}$$

and so with the identification $k = 4$ in (1) and in part **(d)** of Theorem 4.3, we obtain

$$x(t) = \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{4}{s^2 + 16}\right\} + \frac{1}{8}\mathscr{L}^{-1}\left\{\frac{8s}{(s^2 + 16)^2}\right\}$$

$$= \frac{1}{4}\sin 4t + \frac{1}{8}t \sin 4t.$$ ❑



Convolution of Laplace Transform

THEOREM 4.9 Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s).$$

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

and

$$G(s) = \mathscr{L}\{g(t)\} = \int_0^\infty e^{-s\beta} g(\beta)\, d\beta.$$

Proceeding formally, we have

$$F(s)G(s) = \left(\int_0^\infty e^{-s\tau} f(\tau)\, d\tau\right)\left(\int_0^\infty e^{-s\beta} g(\beta)\, d\beta\right)$$

$$= \int_0^\infty \int_0^\infty e^{-s(\tau+\beta)} f(\tau) g(\beta)\, d\tau\, d\beta$$

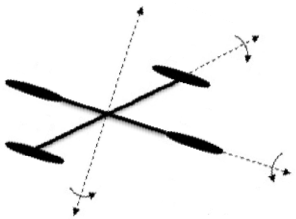
$$= \int_0^\infty f(\tau)\, d\tau \int_0^\infty e^{-s(\tau+\beta)} g(\beta)\, d\beta.$$

Holding τ fixed, we let $t = \tau + \beta$, $dt = d\beta$, so that

$$F(s)G(s) = \int_0^\infty f(\tau)\, d\tau \int_\tau^\infty e^{-st} g(t-\tau)\, dt.$$

In the $t\tau$-plane we are integrating over the shaded region in Figure 4.32. Since f and g are piecewise continuous on $[0, \infty)$ and of exponential order, it is possible to interchange the order of integration:

$$F(s)G(s) = \int_0^\infty e^{-st}\, dt \int_0^t f(\tau) g(t-\tau)\, d\tau = \int_0^\infty e^{-st}\left\{\int_0^t f(\tau) g(t-\tau)\, d\tau\right\} dt = \mathscr{L}\{f * g\}. \quad \square$$

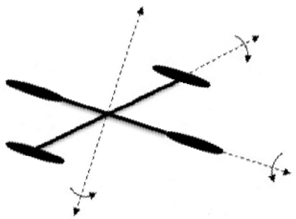


Example 3 **Transform of a Convolution**

Evaluate $\mathscr{L}\left\{\int_0^t e^{\tau} \sin(t-\tau)\, d\tau\right\}$.

SOLUTION With $f(t) = e^t$ and $g(t) = \sin t$ the convolution theorem states that the Laplace transform of the convolution of f and g is the product of their Laplace transforms:

$$\mathscr{L}\left\{\int_0^t e^{\tau} \sin(t-\tau)\, d\tau\right\} = \mathscr{L}\{e^t\} \cdot \mathscr{L}\{\sin t\} = \frac{1}{s-1} \cdot \frac{1}{s^2+1} = \frac{1}{(s-1)(s^2+1)}. \quad \square$$



Example 4 **Inverse Transform as a Convolution**

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}$.

SOLUTION Let

$$F(s) = G(s) = \frac{1}{s^2+k^2}$$

so that

$$f(t) = g(t) = \frac{1}{k}\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \frac{1}{k}\sin kt.$$

In this case (4) gives

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{k^2}\int_0^t \sin k\tau \sin k(t-\tau)\, d\tau. \tag{6}$$

Now recall from trigonometry that

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

and

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Subtracting the first from the second gives the identity

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)].$$

If we set $A = k\tau$ and $B = k(t-\tau)$, we can carry out the integration in (6):

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{2k^2}\int_0^t [\cos k(2\tau - t) - \cos kt]\, d\tau$$

$$= \frac{1}{2k^2}\left[\frac{1}{2k}\sin k(2\tau - t) - \tau\cos kt\right]_0^t$$

$$= \frac{\sin kt - kt\cos kt}{2k^3}.$$

Multiplying both sides by $2k^3$ gives the inverse form of (5). ❑

Transform of an Integral When $g(t) = 1$ and $\mathscr{L}\{g(t)\} = G(s) = 1/s$, the convolution theorem implies that the Laplace transform of the integral of f is

$$\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}. \tag{7}$$

The inverse form of (7),

$$\int_0^t f(\tau)\,d\tau = \mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\}, \tag{8}$$

can be used in lieu of partial fractions when s^n is a factor of the denominator and $f(t) = \mathscr{L}^{-1}\{F(s)\}$ is easy to integrate. For example, we know for $f(t) = \sin t$ that $F(s) = 1/(s^2 + 1)$, and so by (8)

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin\tau\,d\tau = 1 - \cos t$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1 - \cos\tau)\,d\tau = t - \sin t$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (\tau - \sin\tau)\,d\tau = \frac{1}{2}t^2 - 1 + \cos t$$

and so on.

Example 7 **Transform of a Periodic Function**

Find the Laplace transform of the periodic function shown in Figure 4.35.

SOLUTION The function $E(t)$ is called a square wave and has period $T = 2$. On the interval $0 \le t < 2$, $E(t)$ can be defined by

$$E(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2, \end{cases}$$

and outside the interval by $f(t+2) = f(t)$. Now from Theorem 4.10,

$$\mathscr{L}\{E(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} E(t)\, dt = \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} \cdot 1\, dt + \int_1^2 e^{-st} \cdot 0\, dt \right]$$

$$= \frac{1}{1-e^{-2s}} \frac{1-e^{-s}}{s} \qquad \leftarrow 1-e^{-2s} = (1+e^{-s})(1-e^{-s})$$

$$= \frac{1}{s(1+e^{-s})}. \qquad (11)$$